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FINITE ELASTO-PLASTIC DEFORMATION†—I
THEORY AND NUMERICAL EXAMPLES

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Abstract—It is demonstrated that the problem of elasto-plastic finite deformation is governed by a quasi-linear model irrespective of deformation magnitude. This feature follows from the adoption of a rate viewpoint toward finite deformation analysis in an Eulerian reference frame. Objectivity of the formulation is preserved by introduction of a frame-invariant stress rate.

Equations for piece-wise linear incremental finite element analysis are developed by application of the Galerkin method to the instantaneously linear governing differential equations of the quasi-linear model. Numerical solution capability has been established for problems of plane strain and plane stress. The accuracy of the numerical analysis is demonstrated by consideration of a number of problems of homogeneous finite deformation admitting comparative analytic solution. It is shown that accurate, objective numerical solutions can be obtained for problems involving dimensional changes of an order of magnitude and rotations of a full radian.

INTRODUCTION

Many of the presently available formulations for analysis of inelastic finite deformation are primarily intended for analysis of plate and shell problems involving large displacements (rotations) but "small" strains. Consequently these formulations are inappropriate for application to bulky geometries such as occur in forming of metals, necking in tensile bars, and localized deformation of material in the vicinity of stress raisers, especially notches and cracks. Typically such theories are implemented by direct development of linear incremental stiffness equations for finite element analysis in a Lagrangian (or material) reference frame. While this piecewise-linear approach to the problem is appropriate for consideration of elasto-plastic flow[1], its application to geometric non-linearity[2–4] has lead to increasingly complex forms of the incremental equations without providing an indication of the completeness of the formulation. Excellent surveys of this sort of effort are given by Marcal[5] and Stricklin *et al.*[6] and need not be repeated here.

Two exceptions to the above limitations deserve mention. Oden *et al.*[7] have developed analyses for finite hyperelastic deformation, although the nature of the problem precludes direct extension of their techniques to elasto-plastic flow. Hibbitt *et al.*[8] have formulated but not implemented a Lagrangian finite element formulation for the general finite elasto-plastic problem and extracted a small strain–large rotation formulation as a limiting case.

The objectives of our efforts have been to develop governing equations for elasto-plastic

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flow without restricting deformation magnitude, and to establish a numerical solution capability for such problems. The formulation is based upon elasto-plastic constitutive equations generalized from the flow rule appropriate to infinitesimal elasto-plastic deformation of an isotropic work-hardening material. These constitutive equations are distinguished by the introduction of a frame invariant, or objective, stress rate.

The rate form of the constitutive equations suggests that a rate approach be taken toward the entire problem so that flow is viewed as an history dependent process rather than an event. A direct consequence of the consistent adoption of the rate viewpoint in a spatial reference frame is that the problem is found to be governed by quasi-linear differential equations in time and in space. Hence analysis requires solution of an initial- and boundary-value problem involving instantaneously linear equations. The quasi-linear nature of the problem not only suggests an incremental approach to numerical solution but also provides confidence in the completeness of the incremental equations. In the present case finite element solution capability is established; it should be noted, however, that the linearity of the instantaneous governing equations admits use of a wide variety of other established numerical procedures for spatial integration.

In the following sections the initial- and boundary-value problem is formulated, equations for incremental finite element analysis are developed, and numerical solutions are presented for a number of homogeneous finite deformation problems also admitting analytic solution. These results indicate that accurate numerical solutions can be obtained for problems involving dimensional changes of an order of magnitude and rotations of a full radian.

FORMULATION OF THE RATE PROBLEM

Consider the quasi-static deformation depicted in Fig. 1. The total deformation of B_0 into B may be described in a fixed reference frame by the mapping†

$$x^i = x^i(X^I, t) \tag{1}$$

subject to the constraint $|x^i_{,I}| \neq 0$ thereby insuring existence of a unique inverse. In (1) the x^i are spatial coordinates of material particles comprising B at time $t > 0$ which were located

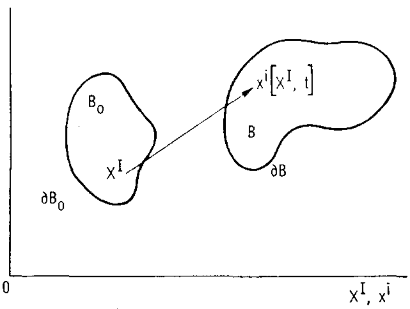

Fig. 1. Deformation mapping.

† The development is presented in general tensor notation. Covariant (contravariant) character is denoted by subscript (superscript) notation. A comma (semicolon) denotes partial (covariant) differentiation. Repeated indices in subscript-superscript pairs indicate summation over the range 1, 2, 3; x^i and X^I are coordinates in a single, fixed, orthogonal curvilinear system; $g_{ij}(G_{IJ})$ is the metric tensor of $x^i(X^I)$; and $\delta_j{}^i$ is the Kronecker delta.

at X^I in B_0 at $t = 0$. Differentiation of (1) with respect to time yields the velocity field v^i which may be viewed as a function of either x^i or X^I. Adopting an Eulerian, or spatial, viewpoint we take $v^i = v^i(x^j, t)$ and may consequently describe the instantaneous rate of deformation by the velocity gradient

$$v_{i;j} = d_{ij} + \omega_{ij}. \tag{2}$$

In (2) the symmetric deformation rate tensor d_{ij} and skew-symmetric spin tensor ω_{ij} are given as

$$d_{ij} = (1/2)(v_{i;j} + v_{j;i}) \quad , \quad \omega_{ij} = (1/2)(v_{i;j} - v_{j;i}). \tag{3}$$

The deformation rate and spin tensors are of particular consequence in that while retaining linear dependence upon the velocity field they provide a full description of continuum motion, within a rigid translation.

Elasto-plastic flow equations

In generalizing the constitutive equations of elasto-plastic flow our intention is to preserve the character of the infinitesimal theory[9] while allowing the total deformation to be viewed as finite. We thus exclude the more general treatments of Lee[10] and of Green and Naghdi [11], and consider the deformation rate tensor d_{ij} to be the sum of recoverable (elastic) and irrecoverable (plastic) components: $d_{ij}{}^{(e)}$ and $d_{ij}{}^{(p)}$, respectively.† It is further assumed that the dependence of $d_{ij}{}^{(e)}$ and of $d_{ij}{}^{(p)}$ upon stress, stress rate, and deformation history may be prescribed independently. The instantaneous plastic, permanently deformed configuration is taken as a stress free reference state for infinitesimal elastic deformation. The generalization of the infinitesimal theory requires precise interpretation of the time rates appearing in the constitutive equations which must satisfy the constraint of objectivity or observer independence.

Plastic flow is presumed incompressible ($d_i^{i(p)} = 0$) and to be governed by two postulates:

(i) There exists a scalar loading function $f(\sigma^{ij}, W^{(p)}) \leq 0$. For convenience, the loading function is taken in the form

$$f = \phi(J_2, J_3) - \kappa(W^{(p)}) \tag{4}$$

in which the yield function ϕ may be viewed as an equivalent stress† τ_{eq} dependent upon the deviatoric stress invariants J_2 and J_3, and κ is a work-hardening parameter dependent solely upon prior plastic work through the plastic strain energy density $W^{(p)}$. As in, for example, Fung's development[9], plastic flow, or loading, requires‡ $f = 0$, $\dot{\phi} > 0$. We may further define cases of neutral loading $\dot{\phi} = 0$ and unloading $\dot{\phi} < 0$.

(ii) Work-hardening plastic flow is constrained by Drucker's hypothesis[12] requiring that positive work be done by an external agent applying and removing a small self-equilibrated stress system $\delta\sigma^{ij}$ which induces plastic flow $\delta\varepsilon_{ij}{}^{(p)}$. The infinitesimal theory is founded upon recognition that Drucker's hypothesis implies

$$\delta\sigma^{ij}\delta\varepsilon_{ij}{}^{(p)} > 0. \tag{5}$$

For finite plastic deformation we consider the inequality (5) in the form

$$\hat{\sigma}^{ij}d_{ij}{}^{(p)} > 0. \tag{6}$$

† (e), (p), eq not indices, throughout.
‡ The · operator denotes a material derivative, e.g. $\dot{f} = \partial f/\partial t + (\partial f/\partial x^i)v^i$

The stress rate $\hat{\sigma}^{ij}$ in (6) is a Jaumann rate[13]

$$\hat{\sigma}^{ij} \equiv \dot{\sigma}^{ij} + \sigma_m{}^i \omega^{mj} - \sigma_m{}^j \omega^{im}. \tag{7}$$

The Jaumann stress rate provides an objective measure of the change in stress viewed from a frame rotating with the material. Its suitability for use in plastic flow equations has been noted by Prager[14]; Hibbitt *et al.*[8] have employed it in a Lagrangian formulation.

The significance of the inequality (6) can be elicited by considering the increment in plastic strain energy density associated with the loading cycle envisioned by Drucker. To terms of the second order in time for an increment δt

$$\delta W^{(p)} \sim [\sigma^{ij} d_{ij}{}^{(p)}]\, \delta t + [\dot{\sigma}^{ij} d_{ij}{}^{(p)} + \sigma^{ij} \dot{d}_{ij}{}^{(p)}](\delta t)^2. \tag{8}$$

The portion of $\delta W^{(p)}$ associated with the work of the external agent is not evident in (8) since in the presence of rotation $\dot{\sigma}^{ij}$ is dependent upon existing stresses. However, $\delta W^{(p)}$ may be rewritten in terms of objective quantities as

$$\delta W^{(p)} \sim \sigma^{ij}[d_{ij}{}^{(p)}\, \delta t + \hat{d}_{ij}{}^{(p)}(\delta t)^2] + \hat{\sigma}^{ij} d_{ij}{}^{(p)}(\delta t)^2. \tag{9}$$

Inspection of (9) reveals that (6) requires that, independent of rotation, the work of the external agent be positive.

Analogous to the implications of Drucker's hypothesis for the infinitesimal case, (6) implies the convexity of the surface $\phi - \kappa = 0$ in stress space, normality of $d_{ij}{}^{(p)}$ to that surface and the existence of a linear relation between $\hat{\sigma}^{ij}$ and $d_{ij}{}^{(p)}$.† As in the infinitesimal case[1] a flow rule is derived from these postulates in the form

$$d_{ij}{}^{(p)} = (\tau_{eq}/2\mu_{eq}{}^{(p)}) \left[\frac{(\partial\tau_{eq}/\partial\sigma^{ij})(\partial\tau_{eq}/\partial\sigma^{kl})}{(\partial\tau_{eq}/\partial\sigma^{rs})\sigma^{rs}}\right] \hat{\sigma}^{kl}. \tag{10}$$

In (10) the interpretation of ϕ as an equivalent stress τ_{eq} permits definition of an equivalent plastic modulus $\mu_{eq}{}^{(p)}$:

$$2\mu_{eq}{}^{(p)} \equiv d\tau_{eq}/d\varepsilon_{eq}{}^{(p)} \tag{11}$$

where $\varepsilon_{eq}{}^{(p)}$ is the time integral of $d_{eq}{}^{(p)}$, and

$$\tau_{eq} d_{eq}{}^{(p)} = \dot{W}^{(p)} = \sigma^{ij} d_{ij}{}^{(p)}. \tag{12}$$

The infinitesimal, isotropic, elastic portion of the deformation is taken as the difference between total and permanent deformation. Consequently the stress and elastic strain may be related by a generalized form of Hooke's law as

$$\varepsilon_{ij}{}^{(e)} = M_{ijkl}\sigma^{kl}. \tag{13}$$

$$M_{ijkl} \equiv (1/2\mu)\{(1/2)(g_{ik}g_{jl} + g_{il}g_{jk}) - [\nu/(1+\nu)]g_{ij}g_{kl}\}. \tag{14}$$

In (14) $2\mu \equiv E/(1+\nu)$, E, and ν are linear elastic constant analogous to the shear and tensile moduli and Poisson's ratio of classical elasticity.

The elastic strains may not be defined *a priori* in terms of problem kinematics but are taken as the time integral of $d_{ij}{}^{(e)}$. Consistent with the objective form of the plastic flow rule (10), Jaumann differentiation of (13) yields

$$d_{ij}{}^{(e)} = M_{ijkl}\hat{\sigma}^{kl}. \tag{15}$$

† The assumption of infinitesimal elastic deformation allows exclusion of concavity in ϕ (see Palmer *et al.* [15]). Corners in ϕ are not excluded from the theory but are not considered here.

The total deformation rate is simply the sum of its elastic and plastic components. Assembling (10) and (15) produces the elasto-plastic flow rule

$$\begin{aligned} 2\mu d_{ij} &= B_{ijkl}\hat{\sigma}^{kl} \\ B_{ijkl} &\equiv (1/2)[g_{ik}g_{jl} + g_{il}g_{jk}] - [\nu/(1+\nu)]g_{ij}g_{kl} \\ &\quad + \gamma^2[(\partial\tau_{\text{eq}}/\partial\sigma^{ij})(\partial\tau_{\text{eq}}/\partial\sigma^{kl})] \\ \gamma^2 &\equiv \tau_{\text{eq}}(\mu/\mu_{\text{eq}}{}^{(\text{p})})/[(\partial\tau_{\text{eq}}/\partial\sigma^{rs})\sigma^{rs}]. \end{aligned} \tag{16}$$

For $\mu_{\text{eq}}{}^{(\text{p})} \neq 0$ (16) may be inverted to give

$$\begin{aligned} \hat{\sigma}^{ij} &= P^{ijkl}d_{kl} \\ P^{ijkl} &\equiv \mu\{(g^{ik}g^{jl} + g^{il}g^{jk}) + [2\nu/(1-2\nu)](g^{ij}g^{kl})\} \\ &\quad - \frac{2\mu(\partial\tau_{\text{eq}}/\partial\sigma^{ab})(\partial\tau_{\text{eq}}/\partial\sigma^{cd})g^{ai}g^{bj}g^{ck}g^{dl}}{1/\gamma^2 + (\partial\tau_{\text{eq}}/\partial\sigma^{mn})(\partial\tau_{\text{eq}}/\partial\sigma^{rs})g^{mr}g^{ns}}. \end{aligned} \tag{17}$$

Note that in the limit $\mu_{\text{eq}}{}^{(\text{p})} \to \infty$ (16, 17) reduce to the elastic equations (15) and its inverse which are then admissible forms irrespective of deformation magnitude, although not necessarily appropriate for any real material.

Rate equilibrium

The rate form of (16, 17) motivates use of equilibrium equations governing the rate of loading upon a deforming solid. For a body in static equilibrium in the absence of body force we have

$$\begin{aligned} &\sigma^{ij}{}_{;j} = 0 \qquad \text{in } B \\ &\sigma^{ij}n_j = t^i \qquad \text{on } \partial B \\ &\int_{\partial B} t^i \,\mathrm{d}S = F^i = 0. \end{aligned} \tag{18}$$

In (18) ∂B is the boundary of B at t, n_j is the unit outer normal to ∂B, t^i is the traction on ∂B, and $F^i = 0$ is the net load on the body.

Preservation of equilibrium in the presence of varying traction on ∂B requires $\dot{F}^i = 0$ or

$$\dot{F}^i = \int_{\partial B} [\dot{t}^i + \sigma^{ij}v^p{}_{;p}n_j]\,\mathrm{d}S = 0 \tag{19}$$

where the traction rate $\dot{t}^i$ is given as

$$\dot{t}^i = (\dot{\sigma}^{ij} - \sigma^{pi}v^j{}_{;p})n_j. \tag{20}$$

Equations (19) imply the rate equilibrium equations

$$\dot{\sigma}^{ij}{}_{;j} - \sigma^{pi}{}_{;j}v^j{}_{;p} = 0 \qquad \text{in } B. \tag{21}$$

Equations (19, 20) also define traction rate boundary conditions on any portion $\partial B' \in \partial B$. If the traction rate is known, as, e.g. in pressurization, (20) apply. If the rate of total load $\dot{F}'^i$ on $\partial B'$ is known then (19) apply so that

$$\dot{F}'^i = \int_{\partial B'} (\dot{t}^i + \sigma^{ij}v^p{}_{;p}n_j)\,\mathrm{d}S. \tag{22}$$

The finite deformation process

The elements of a complete theory are now in hand. The velocity field in a deforming solid is chosen as the principal dependent variable thus guaranteeing flow compatibility. Substituting the constitutive equations (17) into the equilibrium equations (21) we obtain

$$(1/2)[\delta_n{}^i(\delta_m{}^l g^{tk} - \delta_m{}^t g^{lk}) - \delta_m{}^k \delta_n{}^t g^{li} - \delta_n{}^k \delta_m{}^l g^{ti}][\sigma^{mn} v_{t;\,l}]_{;\,i} + [P^{ijkl} v_{j;\,l}]_{;\,i} + \sigma^{kp} v^l{}_{;\,pl} = 0 \quad (23)$$

as the equations governing the instantaneous spatial variation of the velocity field. Equations (23) provide a quasi-linear† model of the entire deformation process and furthermore are linear at any instant of time. Their solution requires combined integration in space and time. Immediate integration of (23) with respect to time, thus defining equations for total deformation, is possible only for homogeneous deformation under proportional loading.

For sufficiently stiff material (σ^{ij}/μ, $\sigma^{ij}/\mu_{eq}{}^{(p)} \ll 1$) undergoing infinitesimal deformation (23) may be specialized to yield the equations of infinitesimal elasto-plastic flow developed by Swedlow[1]. In the limit $\mu_{eq}{}^{(p)} \to \infty$ these equations may in turn be integrated with respect to time yielding the Navier displacement equations of classical elasticity. It should be noted that, while $\sigma^{ij}/\mu \ll 1$ is a reasonable assumption for most metals, the validity of assuming $\sigma^{ij}/\mu_{eq}{}^{(p)} \ll 1$ is dependent upon both material properties and deformation magnitude (see, e.g. Hill[16], Rice[17]).

Complete definition of the finite elasto-plastic deformation problem includes the governing equations (23), prescription of initial and boundary conditions, and specification of material properties. The initial configuration and any initial equilibrated stresses must be known. Admissible boundary conditions include components of traction rate and/or velocity on the deforming boundary. Traction rate, velocity, mixed, and mixed–mixed problems may be defined. The elastic constants of (14) must be known. Work-hardening plastic flow character of the material is defined by the equivalent plastic modulus prescribed as a function of equivalent stress; any history of prior plastic work is reflected in this function.

The governing equations are specialized for analysis of plane stress and plane strain in the Appendix. Other cases involving two spatial coordinates may also be developed following Swedlow[1].

NUMERICAL SOLUTION

The quasi-linear nature of the velocity equilibrium equations suggests the adoption of an incremental approach to numerical integration with respect to time. The availability of the field formulation provides assurance of the completeness of the incremental equations and allows the use of any convenient procedure for spatial integration over the domain B. In the present instance the choice has been made in favor of a simple first order expansion in time for the construction of incremental solutions from the results of finite element spatial integration of the governing equations.

The procedure employed permits the rates of the field formulation to be interpreted as increments in the numerical solution. This is particularly convenient for the construction of incremental boundary condition histories. For consistency and clarity, however, the rate notation is preserved in the following discussion.

† Elastic unloading introduces non-linearity through the dependence of P^{ijkl} on stress rate though $\dot{\varphi}$, i.e. $\mu_{eq}{}^{(p)}$ is given by (11) for $\dot{\varphi} > 0$ but is unbounded for $\dot{\varphi} \leq 0$.

Spatial integration

The finite element method for spatial discretization of continuum problems has been well documented (see, e.g. Zienkiewicz[18] or Oden[7]) and will not be detailed here. It should be noted, however, that as a consequence of the introduction of the Jaumann stress rate, the velocity equilibrium equations (23) are not symmetric. This feature precludes implementation of a Ritz procedure as commonly employed in finite element analysis of infinitesimal deformation. Linear algebraic equations governing the discrete model for the finite case are developed employing the method of Galerkin[19]. The equations are given in general three-dimensional form and may be specialized to particular problems in any orthogonal coordinate system.

Restricting attention to a single finite element B_m, the velocity field is approximated† by $\tilde{v}_i(x^j)$.

$$v_i(x^j) \sim \tilde{v}_i(x^j) = \overset{T}{\Gamma^{\alpha\beta}} \psi_i{}^{\alpha}(x^j) V^{\beta} \qquad \alpha, \beta = 1, \ldots, N. \tag{24}$$

In (24) V^{β} are generalized nodal velocities, $\Gamma^{\alpha\beta}$ is dependent upon the present nodal coordinates, $\psi_i{}^{\alpha}(x^j)$ is a vector of prescribed functions of x^j, and N is the number of degrees of freedom associated with the element. The matrix $\Gamma^{\alpha\beta}$ is defined by requiring that evaluation of $\tilde{v}_i$ at nodal positions yield V^{β}.

The Galerkin method is based on the observation that if the $\psi_i{}^{\alpha}$ in (24) are considered independent, then requiring $\tilde{v}_i$ to satisfy (23) as N tends to infinity implies orthogonality of each of $\psi_i{}^{\alpha}$ to (23) in B_m. Thus

$$\int_{B_m} [L^i(\tilde{v}_j)] \psi_i{}^{\alpha} \, \mathrm{d}V = 0 \qquad \alpha = 1, \ldots, N \tag{25}$$

in which $L^i(\tilde{v}_j)$ represents (23) in terms of (24). Expansion of (25) for finite N yields approximate rate stiffness equations for the element B_m:

$$\int_{\partial B_m} \left[\dot{t}_i \overset{T}{\Gamma^{\eta\beta}} \psi^{\beta i} + t_i \overset{T}{\Gamma^{\eta\beta}} \psi^{\beta i} \overset{T}{\psi^{\delta p}}{}_{;p} \right] \Gamma^{\delta\alpha} V^{\alpha} \, \mathrm{d}S = K^{\eta\alpha} V^{\alpha} \tag{26}$$

$$K^{\eta\alpha} \equiv \int_{B_m} \left\{ \overset{T}{\Gamma^{\eta\beta}} \left[\psi^{\beta}_{ij} P^{ijkl} \overset{T}{\psi^{\delta}_{kl}} - 2\psi^{\beta}_{ij} \sigma^{j}{}_{m} \overset{T}{\psi^{\delta im}} + \psi^{\beta}_{j;\,i} \sigma^{ip} \overset{T}{\psi^{\delta j}_{;\,p}} + \psi^{\beta}_{i;\,p} \sigma^{ip} \overset{T}{\psi^{\delta j}_{;\,j}} \right] \Gamma^{\delta\alpha} \right\} \mathrm{d}V$$

$$\psi_{ij}{}^{\alpha} \equiv (1/2)(\psi^{\alpha}_{i;\,j} + \psi^{\alpha}_{j;\,i}). \tag{27}$$

The surface integral in (26) corresponds to the time rate of total load (22) on the element boundary and defines a vector of generalized nodal load rates T^{α} corresponding to V^{α}. Hence (26) may be written as

$$T^{\eta} = K^{\eta\alpha} V^{\alpha} \qquad \eta, \alpha = 1, \ldots, N. \tag{28}$$

The element rate stiffness matrix $K^{\eta\alpha}$ is full, depends upon the deformed geometry through the nodal coordinates in $\Gamma^{\alpha\beta}$ and upon the existing state of stress. In general $K^{\eta\alpha}$ is not symmetric.

Having written (28) for each element, master rate stiffness equations are written for the entire body by summing load rate components at each node. Thus

$$T^{\eta} = \boldsymbol{K}^{\eta\alpha} V^{\alpha} \qquad \eta, \alpha = 1, \ldots, M \tag{29}$$

† Greek superscripts denote matrix character associated with nodal variables. Repeated Greek superscripts indicate summation over the range of the indices. Overscript T denotes the transpose of a matrix.

where M is the total number of degrees of freedom associated with the finite element model. The matrix $\boldsymbol{K}^{\eta\alpha}$ is $M \times M$, sparse and not symmetric. Banded coefficient structure may be achieved in $\boldsymbol{K}^{\eta\alpha}$ by appropriate construction of the map of elements.

As with other finite element methods solution of (29) requires specification of M of the $2M$ variables, T^α and V^α. At internal nodes equilibrium requires T^α to be zero. At boundary nodes either T^α or V^α must be prescribed for each degree of freedom.

Incremental analysis

The solution to (29) at time t_0 provides a basis for evaluation of a deformation increment and associated changes in internal stresses and boundary loading. The incremental solution defines the deformed configuration and stress state at $t = t_0 + \delta t$ thereby permitting definition of a new spatial problem at the later time.†

Nodal quantities at t may be approximated directly. The nodal coordinates are found as

$$x^\alpha(t) \sim x^\alpha(t_0) + V^\alpha\, \delta t. \tag{30}$$

Similarly, the nodal loads are given by

$$F^\alpha(t) \sim F^\alpha(t_0) + T^\alpha\, \delta t. \tag{31}$$

Evaluation of element stresses at t requires explicit consideration of the change in reference state produced by the deformation increment. Denoting the Cauchy stress at t_0 as $\mathring{\sigma}^{ij}$ we define a symmetric Kirchhoff stress S^{ij} such that

$$\lim_{t \to t_0} [S^{ij}(\mathring{x}^k, t)] = \sigma^{ij}(\mathring{x}^k, t_0) = \mathring{\sigma}^{ij}(\mathring{x}^k) \tag{32}$$

where $\mathring{x}^k$ are the element field coordinates at t_0. The Cauchy stress at t is therefore given as

$$\sigma^{ij}(\tilde{x}^k, t) = (1/J)(\partial\tilde{x}^i/\partial\mathring{x}^k)(\partial\tilde{x}^j/\partial\mathring{x}^l)S^{kl}(\mathring{x}^r, t) \tag{33}$$

in which $\tilde{x}^i$ are the element field coordinates at t and $J = (\tilde{g}/\mathring{g})|\partial\tilde{x}^i/\partial\mathring{x}^j|$, where $\tilde{g}$ and $\mathring{g}$ are determinants of the metric tensors of $\tilde{x}^i$ and $\mathring{x}^i$, respectively.

$$S^{kl}(t) \sim \mathring{\sigma}^{kl} + \dot{S}^{kl}\, \delta t. \tag{34}$$

In (34) the Kirchhoff stress rate may be computed from the Jaumann stress rate at t_0 yielding

$$\dot{S}^{ij} = \hat{\sigma}^{ij} + \mathring{\sigma}^{ij}d_k{}^k - \mathring{\sigma}^{jp}d_p{}^i - \mathring{\sigma}^{ip}d_p{}^j \tag{35}$$

where $\hat{\sigma}^{ij}$ is given by (17).

The incremental deformation gradient $\partial\tilde{x}^i/\partial\mathring{x}^j$ is found by approximating the incremental deformation as the mapping

$$\tilde{x}^i \sim \mathring{x}^i + u^i(\mathring{x}^k). \tag{36}$$

The displacement field $u^i(\mathring{x}^k)$ in (36) is represented in the form

$$u^i(\mathring{x}^k) = \overset{T}{\Gamma^{\alpha\beta}}u^\beta\psi^{\alpha i}(\mathring{x}^k) \tag{37}$$

in which $u^\beta = V^\beta\, \delta t$ are incremental nodal displacements and $\psi^{\alpha i}$ are defined in (24).

† Should elastic unloading occur during an increment, P^{ijkl} must be appropriately recomputed and the analysis for that increment repeated. While convergence of this process has not been proven[22], operational experience indicates that it is well behaved.

The total deformation gradient, and consequently any of the total finite strain tensors, may be computed in a similar manner by considering total rather than incremental displacements in (36, 37).

NUMERICAL EXAMPLES

The foregoing formulation has been implemented for analysis of finite deformation under conditions of plane stress or plane strain (see the Appendix). Verification of the computer program FIPDEF† [23, 24] requires consideration of:

(i) completeness of the theoretical and numerical formulation,
(ii) numerical error (round-off),
(iii) temporal and spatial discretization error, and
(iv) the range of applicability of the constitutive formulation for particular materials.

The present discussion is limited to problems of homogeneous deformation admitting analytic solution and thereby permitting evaluation of (i) and (ii) above. The analysis has also been employed for the study of tensile necking[20]. Evaluation of these results and discussion of (iii) and (iv) are reported separately[21].

Finite extension

Consider the undeformed cube of Fig. 2a. The cube is homogeneously deformed into a bar (Fig. 2b), under axial load ($\sigma_x > 0$, $\sigma_y = 0$). The deformed geometry is fully described by the stretch ratios

$$\lambda_x \equiv l_x/l_0,\ \lambda_y \equiv l_y/l_0,\ \lambda_z \equiv l_z/l_0. \tag{38}$$

The deformation may proceed under conditions of plane stress ($\sigma_z = 0$) or plane strain ($\lambda_z = 1$); in either case the applied load is found as

$$P_x = \sigma_x A_x \tag{39}$$

where we note that: $\sigma_x = \sigma_x(\varepsilon_x)$, $\varepsilon_x = \ln \lambda_x$, $A_x = l_0{}^2 \lambda_y \lambda_z$.

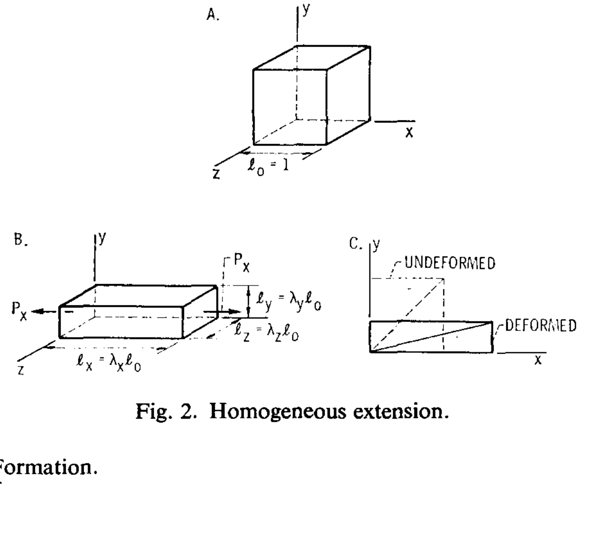

Fig. 2. Homogeneous extension.

† FInite Plastic DEFormation.

We consider finite extension of two types of material, elastic and bilinear elasto-plastic. To verify the elastic analysis equations (15) are employed irrespective of deformation magnitude. The strain hardening character of the elasto-plastic material considered is described by

$$\varepsilon_{ef}^{(p)} = \begin{cases} 0; & \sigma_{ef} < \sigma_Y \\ 1/\beta(\sigma_{ef} - \sigma_Y); & \sigma_{ef} \geq \sigma_Y \end{cases} \tag{40}$$

in which the equivalent stress of (18) has been taken as the effective stress:

$$\sigma_{ef} \equiv [(3/2)s_{ij}s^{ij}]^{1/2} \tag{41}$$

$$s^{ij} \equiv \sigma^{ij} - (1/3)g^{ij}\sigma_l^{\,l}. \tag{41}$$

Analytic solutions for plane stress and plane strain extension of the above materials are developed by Osias[20] through direct integration of the constitutive equations. Note that for homogeneous deformation the equilibrium equations (23) are satisfied identically. The elastic solutions are exact. The elasto-plastic solutions require the assumption of proportional loading. This assumption, valid in plane stress, is not accurate in plane strain since σ_z/σ_x varies from Poisson's ratio in the elastic range to 1/2 for fully plastic flow. The error is, however, negligible for the large, homogeneous deformations considered here.

The essential features of the extension problem are evident in Fig. 3 showing FIPDEF results for applied load, axial stress, and area reduction as functions of axial stretch for elastic plane stress extension. The numerical prediction of nonlinear, multi-valued load-stretch response is within 1 per cent of the analytic solution over the full range of deformation considered.

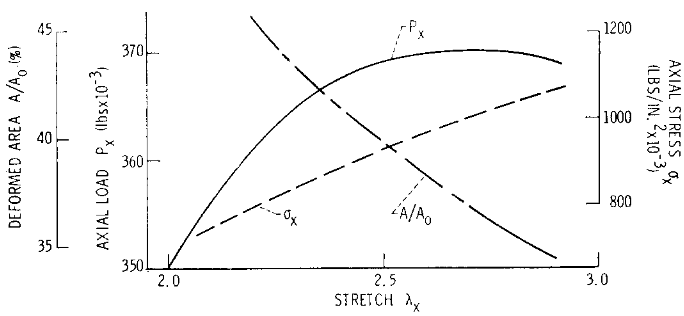

Fig. 3. Finite elastic simple extension.

The maximum load effect can be predicted[25] by differentiation of (39) and solution of the resulting equation for a critical stretch. That is,

$$\dot{P}_x = \dot{\sigma}_x A_x + \sigma_x \dot{A}_x = 0.$$

Analytic and numerical results for maximum load, P_c, and corresponding values of critical stretch, λ_c, and stress, σ_c, are given in Tables 1 and 2 for plane stress and plane strain extension of elastic and elasto-plastic materials, respectively. The effect of elastic Poisson's ratio is shown to illustrate the resolution attainable in the numerical analysis. Load–stretch results for the above cases are given in Figs. 4 and 5. FIPDEF and analytic results agree within 1 per cent.

Table 1. Homogeneous extension of elastic† bodies
Comparison of FIPDEF and analytic‡ results at maximum load

Poisson's ratio	P_c (lb × 10^{-5})		λ_c		σ_c (lb/in² × 10^{-6})	
Plane stress:						
	FIPDEF	analysis	FIPDEF	analysis	FIPDEF	analysis
0·3	6·15	6·13	5·27	5·30	1·67	1·67
0·4	4·62	4·60	3·48	3·49	1·25	1·25
0·5	3·70	3·68	2·71	2·72	1·00	1·00
Plane strain:						
0·3	9·46	9·43	10·16	10·29	2·57	2·56
0·4	6·59	6·57	4·46	4·48	1·79	1·79
0·5§	4·93	4·91	2·71	2·72	1·34	1·34

† Note: $E = 10^6$ lb/in² in all cases.
‡ Ref.[20].
§ Poisson's ratio of 0·5 is inadmissible in plane strain analysis. Since it corresponds to elastic incompressibility the present analyses were performed using a high ratio of bulk modulus to Young's modulus $K/E = 10^3$, which approximates such material.

Table 2. Homogeneous extension of elasto-plastic† bodies
Comparison of FIPDEF and analytic‡ results at maximum load

Poisson's ratio	P_c (lb × 10^{-4})		λ_c		σ_c (lb/in² × 10^{-3})	
Plane stress:						
	FIPDEF	analysis	FIPDEF	analysis	FIPDEF	analysis
0·3	7·78	7·72	1·27	1·27	94·74	94·20
0·4	7·65	7·58	1·25	1·24	93·29	92·56
0·5	7·52	7·44	1·23	1·22	91·45	90·91
Plane strain:						
0·3	9·29	9·17	1·44	1·43	127·6	127·1
0·4	9·20	9·09	1·42	1·41	125·1	124·0
0·5§	9·03	8·92	1·36	1·36	121·7	121·0

† Note: $E = 10^6$ lb/in²
$\beta = d\sigma_{ef}/d\bar{\varepsilon}^{(p)} = 10^5$ for $\sigma_{ef} \geq \sigma_Y$
$\sigma_Y = 8 \times 10^4$ lb/in².
‡ Ref.[20].
§ Poisson's ratio of 0·5 is inadmissible in plane strain analysis. Since it corresponds to elastic incompressibility the present analyses were performed using a high ratio of bulk modulus to Young's modulus $K/E = 10^3$, which approximates such material.

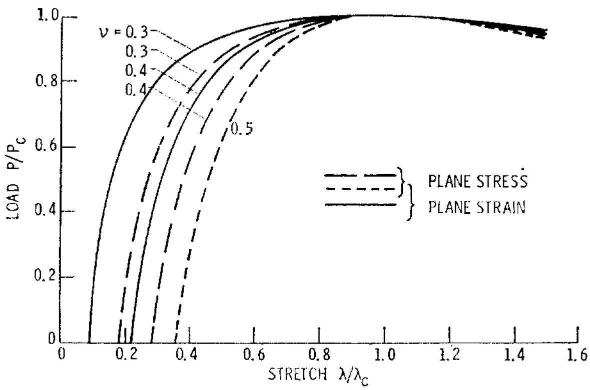

Fig. 4. Load–stretch response for finite elastic extension.

All of the preceding numerical analysis employed the two element map of Fig. 2c which is sufficient for homogeneous deformation. Extension was prescribed in increments of 1 per cent of the original unit length. Thus the number of increments per problem varied from approximately 50 $(1{\cdot}0 \leq \lambda \leq 1{\cdot}5)$ for elasto-plastic problems to 1000 $(1{\cdot}0 \leq \lambda \leq 11{\cdot}0)$ for elastic plane strain extension. No accumulation of error was noted in any problem.

Finite simple shear

The unit cube of Fig. 6a is deformed in planar simple shear by prescribing the velocity field to be

$$v_x = 2ky \tag{42}$$
$$v_y = v_z = 0. \tag{42}$$

The deformed configuration is shown in Fig. 6b and is completely defined by the shear angle γ

$$\gamma = \tan^{-1}\zeta,\ \zeta \equiv 2kt.$$

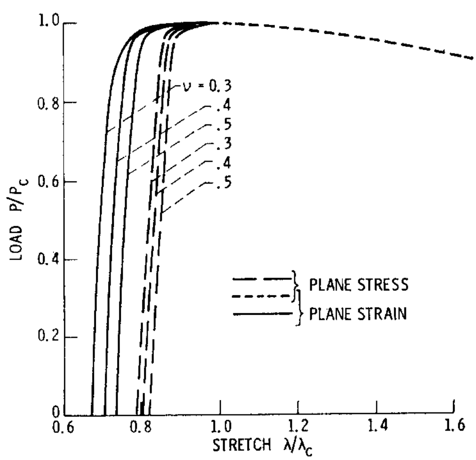

Fig. 5. Load–stretch response for finite elasto-plastic extension.

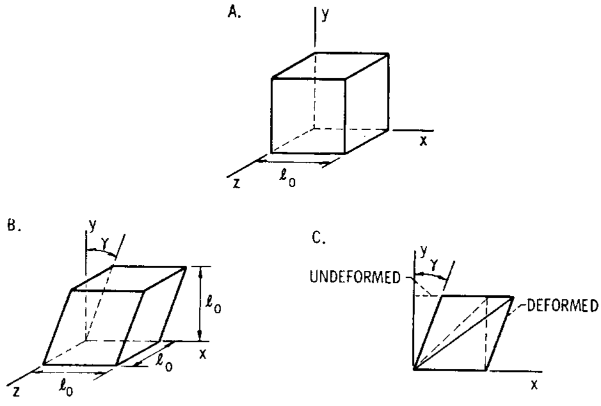

Fig. 6. Finite simple shear.

For an elastic material the stress solution is found by substitution of (42) into (15) and integrating over time. In either plane stress or plane strain the non-zero stress components are:

$$\sigma_{xy} = \mu \sin \zeta$$
$$\sigma_x = \mu(1 - \cos \zeta)$$
$$\sigma_y = \mu(\cos \zeta - 1)$$

where μ is the elastic shear modulus.

The FIPDEF solution employs the two element map of Fig. 6c. Shearing deformation was prescribed in increments of $\delta\zeta = 0{\cdot}01$. Numerical and analytic stress results are compared in Fig. 7. Shear stress is numerically predicted within 1 per cent over the range $0{\cdot}0 \leq \zeta \leq 0{\cdot}7$. Normal stress magnitude increasingly lags the correct value as τ increases building to 5 per cent at $\zeta = 0{\cdot}7$, a shear angle of approximately 35°.

Combined extension and rotation

The introduction of the Jaumann stress rate in the constitutive equations and consequent preservation of objectivity is fundamental to the present formulation. In order to evaluate

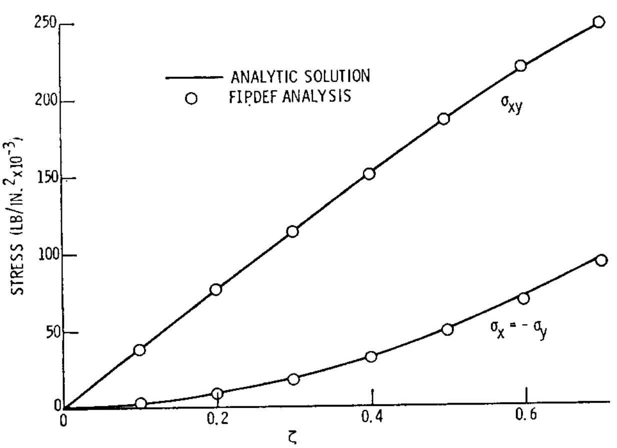

Fig. 7. Stress response for finite elastic simple shear.

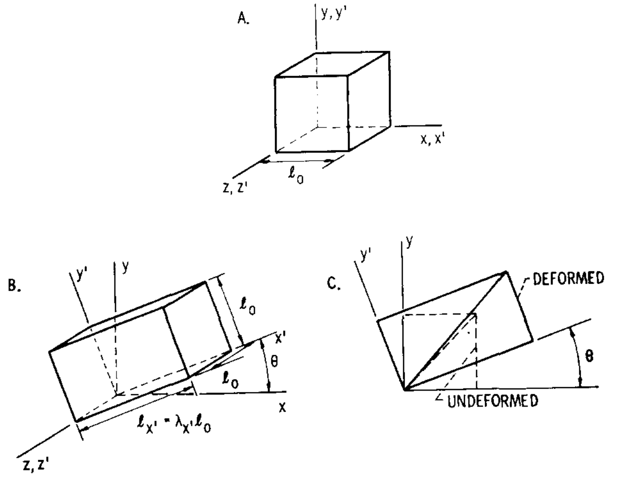

Fig. 8. Finite extension and rotation.

the objective character of the numerical analysis the problem of combined extension and planar rotation of a power law hardening elasto-plastic material has been considered. A particular case of plane stress extension is considered here and depicted in Fig. 8. Viewed from the rotating, primed, coordinate system of that figure the body extends in the x' direction, is of fixed y' dimension and contracts in the z, or plane stress, direction. The effective plastic property relation employed is given by

$$\varepsilon_{ef}^{(p)} = \begin{cases} 0; & \sigma_{ef} < \sigma_Y \\ A[(\sigma_{ef}/\sigma_Y) - 1]^N; & \sigma_{ef} \geq \sigma_Y. \end{cases} \tag{43}$$

Objectivity requires that the fixed (unprimed) frame and rotating (primed) frame solutions to this problem be tensorially equivalent. Both solutions have been generated using the FIPDEF program. In specifying the incremental displacement boundary conditions of the

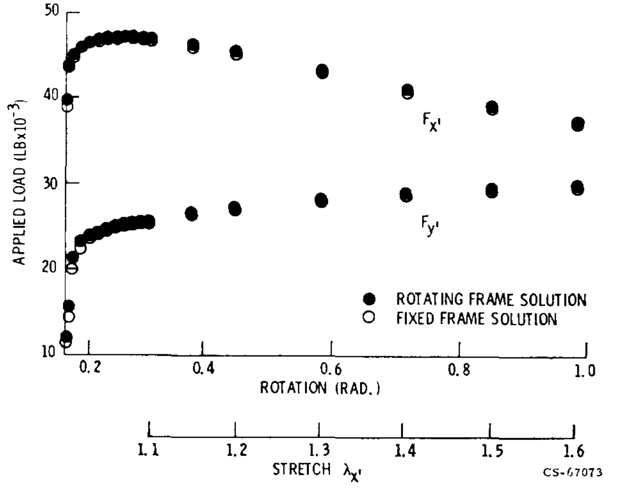

Fig. 9. Load–stretch response for finite elasto-plastic extension and rotation.

two problems the rotating frame solution considers $\theta(t) = 0$ while the fixed frame analysis considers $\theta(t) \geq 0$; the two numerical problems were otherwise identical. Incremental extensions were initially 0·001 and increased according to the algorithm $\delta l_{x'}/(l_{x'} - l_0) = 0{\cdot}05$. Incremental rotations in the fixed frame analysis were 0·00667 radians for all increments.

Figure 9 compares the rotating frame load–stretch solution and the tensor transform of the fixed frame solution. Applied load predictions agree within 1 per cent. Figure 10 compares rotating and fixed frame solution principal stress components.

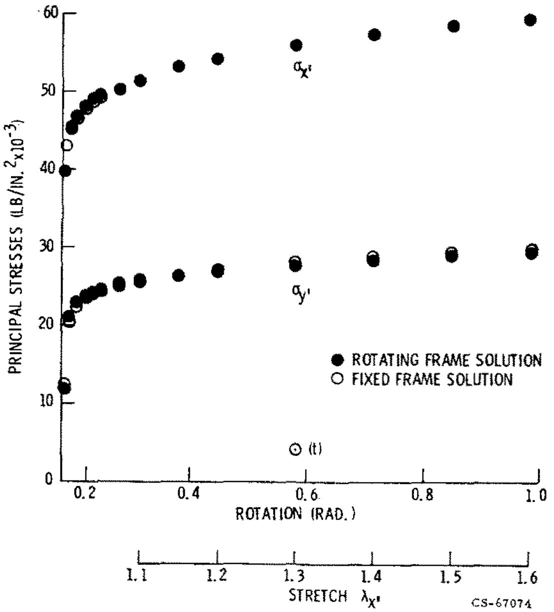

Fig. 10. Stress–stretch response for finite elasto-plastic extension and rotation.

The results demonstrate that the analysis is objective and further that accurate numerical solutions can be obtained for problems involving rotations of a full radian. No appreciable accumulation of equilibrium error was observed over the range of deformation and rotation considered.

CONCLUSIONS

It has been demonstrated that by adoption of a rate viewpoint in an Eulerian reference frame, elasto-plastic deformation may be described by a quasi-linear model irrespective of total deformation magnitude. The problem is posed in the form of an objective initial- and boundary-value problem whose solution requires integration in time and space. The formulation reduces to well established forms in the limits of infinitesimal elastic and elasto-plastic deformation.

The quasi-linear nature of the governing equations motivates the choice of an incremental approach to numerical integration with respect to time and further, guarantees the completeness of the *linear* incremental equations. Spatial integration is accomplished using a finite element approach. The Eulerian reference frame of the analysis allows the deformed states of a body to be represented by simply up-dating nodal coordinates.

The completeness and accuracy of a two-dimensional finite element formulation have been demonstrated by solution of problems of simple, homogeneous, but highly non-linear, elastic and elasto-plastic finite deformation. The results indicate that accurate numerical solutions may be obtained for problems involving dimensional changes of an order of magnitude and rotations of a full radian.

The numerical solution capability which has been established provides a basis for study of finite deformation effects in materials testing and metal forming. The simplicity of the numerical formulation provides a basis for introduction of finite deformation solution capability into existing finite element programs originally designed for analysis of infinitesimal deformation.

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APPENDIX

We treat plane strain and plane stress together as their governing equations are similar. In both cases attention is restricted to loading and deformation symmetric with respect to the x_1–x_2 plane in which the analysis is performed. Adopting cartesian notation the independent variables are x, y, and t. The principal dependent variables are velocity components v_x and v_y. The development is restricted to initially stress-free bodies.

Plane strain is defined by the condition $\lambda_z = 1$ for which it is sufficient to consider $d_{zz} = 0$ since the requirement of symmetry with respect to the x–y plane implies $v_z = 0$, $d_{xz} = d_{yz} = 0$ and further that $\partial/\partial z$ is a null operator. Consequently the constitutive stress rate equations (17) reduce to:

$$\begin{aligned} \hat{\sigma}_{xx}/E &= b_{11}d_{xx} + 2b_{12}d_{xy} + b_{13}d_{yy} \\ \hat{\sigma}_{xy}/E &= b_{21}d_{xx} + 2b_{22}d_{xy} + b_{23}d_{yy} \\ \hat{\sigma}_{yy}/E &= b_{31}d_{xx} + 2b_{32}d_{xy} + b_{33}d_{yy} \end{aligned} \tag{44}$$

and also provide the results $\sigma_{xy} = \sigma_{yz} = 0$ and

$$\dot{\sigma}_{zz}[1 + (1+\nu)\gamma^2\tau_{zz}^{\ 2}] = [\nu - (1+\nu)\gamma^2\tau_{xx}\tau_{zz}]\hat{\sigma}_{xx} + [\nu - (1+\nu)\gamma^2\tau_{yy}\tau_{zz}]\hat{\sigma}_{yy} - 2(1+\nu)\gamma^2\tau_{xy}\tau_{zz}\hat{\sigma}_{xy}. \tag{45}$$

The coefficients $b_{ij} = b_{ji}$ are found as:

$$\begin{aligned} b_{11} &= (\lambda + 2\mu)/E - \gamma^2\tau_{xx}^{\ 2}/\bar{P} \\ b_{12} &= -\gamma^2\tau_{xx}\tau_{xy}/\bar{P} \\ b_{13} &= \lambda/E - \gamma^2\tau_{xy}\tau_{yy}/\bar{P} \\ b_{22} &= \mu/E - \gamma^2\tau_{xy}^{\ 2}/\bar{P} \\ b_{23} &= -\gamma^2\tau_{xy}\tau_{yy}/\bar{P} \\ b_{33} &= (\lambda + 2\mu)/E - \gamma^2\tau_{yy}^{\ 2}/\bar{P} \end{aligned} \tag{46}$$

in which $\tau_{ij} = \partial\tau_{eq}/\partial\sigma^{ij}$, γ^2 is defined in (16) and

$$\bar{P} = (1+\nu)[1 + \gamma^2(\tau_{xx}^{\ 2} + \tau_{yy}^{\ 2} + \tau_{zz}^{\ 2} + 2\tau_{xy}^{\ 2})].$$

Plane stress is defined by requiring $\sigma_z = \sigma_{xz} = \sigma_{yz} = 0$ which in conjunction with the planar symmetry constraint implies (employing (7))

$$\hat{\sigma}_z = \hat{\sigma}_{xz} = \hat{\sigma}_{yz} = 0. \tag{47}$$

We also find from symmetry that $v_z = 0$ in the x–y plane. The constitutive flow equations (16) provide the results $d_{xz} = d_{yz} = 0$ and

$$Ed_{zz} = [-\nu + (1+\nu)\gamma^2\tau_{xx}\tau_{zz}]\hat{\sigma}_{xx} + [-\nu + (1+\nu)\gamma^2\tau_{yy}\tau_{zz}]\hat{\sigma}_{yy} + [2(1+\nu)\gamma^2\tau_{zz}\tau_{xy}]\hat{\sigma}_{xy} \tag{48}$$

and the stress rate equations again take the form (44).

The b_{ij} are now given by

$$\begin{aligned} b_{11} &= 1/(1-\nu^2) - \gamma^2(\tau_{xx} + \nu\tau_{yy})^2/(1-\nu)P \\ b_{12} &= -\gamma^2(\tau_{xx} + \nu\tau_{yy})\tau_{xy}/P \\ b_{13} &= \nu/(1-\nu^2) - \gamma^2(\tau_{xx} + \nu\tau_{yy})(\tau_{yy} + \nu\tau_{xx})/(1-\nu)P \\ b_{22} &= 1/[2(1+\nu)] - (1-\nu)\gamma^2\tau_{xy}^{\ 2}/P \\ b_{23} &= -\gamma^2(\tau_{yy} + \nu\tau_{xx})\tau_{xy}/P \\ b_{33} &= 1/(1-\nu^2) - \gamma^2(\tau_{yy} + \nu\tau_{xx})^2/(1-\nu)P \\ P &= (1+\nu)\{1 - \nu + \gamma^2[(\tau_{xx} + \tau_{yy})^2 + 2(1-\nu)(\tau_{xy}^{\ 2} - \tau_{xx}\tau_{yy})]\}. \end{aligned} \tag{49}$$

Note that d_{zz} in (48) may be taken as $\dot{\lambda}_z/\lambda_z$ where λ_z is a thickness averaged stretch indicating a variation in thickness with continuing in-plane deformation which must be reflected in the traction boundary conditions (19, 20).

Employing (3, 7, and 21) the velocity equilibrium equations (23) for the planar cases take the form (50) in which the b_{ij} are given by (46) or (49) for plane strain or plane stress, respectively.

$$
\begin{aligned}
&b_{11}v_{x,xx} + (2b_{12} + \sigma_{xy}')v_{x,xy} + [b_{22} + (1/2)(\sigma_{yy}' - \sigma_{xx}')]v_{x,yy} + (b_{12} - \sigma_{xy}')v_{y,xx} \\
&\quad + [b_{13} + b_{22} + (1/2)(\sigma_{xx}' - \sigma_{yy}')]v_{y,xy} + b_{23}v_{y,yy} + (b_{11,x} + b_{12,y} - \sigma'_{xx,x})v_{x,x} \\
&\quad + (b_{13,x} + b_{23,y} - \sigma'_{xy,y})v_{y,y} + [b_{12,x} + b_{22,y} - (1/2)(\sigma'_{xx,y} + \sigma'_{xy,x})](v_{x,y} + v_{y,x}) = 0. \\
&b_{33}v_{y,yy} + (2b_{23} + \sigma_{xy}')v_{y,yx} + [b_{22} + (1/2)(\sigma_{xx}' - \sigma_{yy}')]v_{y,xx} + (b_{23} - \sigma_{xy}')v_{x,yy} \\
&\quad + [b_{13} + b_{22} + (1/2)(\sigma_{yy}' - \sigma_{xx}')]v_{x,yx} + b_{12}v_{x,xx} + (b_{33,y} + b_{23,x} - \sigma'_{yy,y})v_{y,y} \\
&\quad + (b_{13,y} + b_{12,x} - \sigma'_{xy,x})v_{x,x} + [b_{23,y} + b_{22,x} - (1/2)(\sigma'_{yy,x} + \sigma'_{xy,y})](v_{x,y} + v_{y,x}) = 0.
\end{aligned} \tag{50}
$$

In (50), $\sigma_{ij}' = \sigma_{ij}/E$, $\sigma'_{ij,k} = \sigma_{ij,k}/E$, and $v_{x,x} = \partial v_x/\partial x$ and so forth.

For completeness, we note that the two-dimensional constitutive flow equations (16) take the form (51) for both plane strain and plane stress. The coefficients $a_{ij} = a_{ji}$.

$$
\begin{aligned}
Ed_{xx} &= a_{11}\hat{\sigma}_{xx} + a_{12}\hat{\sigma}_{xy} + a_{13}\hat{\sigma}_{yy} \\
2Ed_{xy} &= a_{21}\hat{\sigma}_{xx} + a_{22}\hat{\sigma}_{xy} + a_{23}\hat{\sigma}_{yy} \\
Ed_{yy} &= a_{31}\hat{\sigma}_{xx} + a_{32}\hat{\sigma}_{xy} + a_{33}\hat{\sigma}_{yy}.
\end{aligned} \tag{51}
$$

In plane strain:

$$
\begin{aligned}
a_{11} &= (1 + \nu)\{1 - \nu + \gamma^2[\tau_{yy}^2 - 2(1 - \nu)\tau_{xx}\tau_{zz}]\}/[1 + (1 + \nu)\gamma^2\tau_{zz}^2] \\
a_{12} &= 2(1 + \nu)\gamma^2\tau_{xy}(\tau_{xx} + \nu\tau_{zz})/[1 + (1 + \nu)\gamma^2\tau_{zz}^2] \\
a_{13} &= (1 + \nu)[-\nu + \gamma^2(\tau_{xx}\tau_{yy} - 2\nu\tau_{zz}^2)]/[1 + (1 + \nu)\gamma^2\tau_{zz}^2] \\
a_{22} &= 2(1 + \nu)\{1 + 2\gamma^2\tau_{xy}^2/[1 + (1 + \nu)\gamma^2\tau_{zz}^2]\} \\
a_{23} &= 2(1 + \nu)\gamma^2\tau_{xy}(\tau_{yy} + \nu\tau_{zz})/[1 + (1 + \nu)\gamma^2\tau_{zz}^2] \\
a_{33} &= (1 + \nu)\{1 - \nu + \gamma^2[\tau_{xx}^2 - 2(1 - \nu)\tau_{yy}\tau_{zz}]\}/[1 + (1 + \nu)\gamma^2\tau_{zz}^2].
\end{aligned}
$$

In plane stress:

$$
\begin{aligned}
a_{11} &= 1 + (1 + \nu)\gamma^2\tau_{xx}^2 & a_{22} &= 2(1 + \nu)(1 + 2\gamma^2\tau_{xy}^2) \\
a_{12} &= 2(1 + \nu)\gamma^2\tau_{xx}\tau_{xy} & a_{23} &= 2(1 + \nu)\gamma^2\tau_{xy}\tau_{yy} \\
a_{13} &= -\nu + (1 + \nu)\gamma^2\tau_{xx}\tau_{yy} & a_{33} &= 1 + (1 + \nu)\gamma^2\tau_{yy}^2.
\end{aligned}
$$

Резюме — Демонстрируется, что проблема финитной упругопластической деформации регулируется квазилинейной моделью независимо от размера деформации. Это становится понятным, если провести анализ финитной деформации по системе отсчета Еюлерия. Объективность формуляции сохраняется введением системы неизменной интенсивности напряжения.

Методом Галеркина развили уравнения анализа кусочного линейного конечного приращения элемента. Установили возможность численного решения для плоской деформации и для напряжения в плоском состоянии. Точность численного анализа демонстрируется принятием во внимание нескольких проблем однородной финитной деформации, допускающей относительное аналитическое решение. Показано, что можно получить точные, объективные численные решения проблем, включающих изменения размера значительной величины и полный поворот радиана.